

The Laplace Transform of the Linear Car-following Model

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Abstract. Laplace transform is used to analyze the differential-difference equation for the linear car-following model. The car-following model has been developed to describe the dynamics of a platoon of vehicles initially traveling at uniform speed and spacing when the lead vehicle introduces a perturbation by decelerating or accelerating. Shifting theorem and the convolution theorem for Laplace transforms are used to model fluctuations in velocities and spacing of following vehicles as a function of the deceleration or acceleration of the lead car when the initial velocity and spacing are given.

1. Introduction

The car-following model has been developed to describe the dynamics of a platoon of vehicles initially traveling at uniform speed and spacing when the lead vehicle introduces a perturbation by braking or accelerating. The linear car-following model was developed by Chandler, Herman and Montroll [1]. Assume that $x_n(t)$ gives position of the n^{th} car in the platoon at time t , and $v_n(t)$ is the corresponding velocity function. Thus the position and velocity functions for the leading car are given by $x_0(t)$ and $v_0(t)$, respectively. Chandler, Herman and Montroll assume that the acceleration of the n^{th} car with time delay T is proportional to the relative velocities of the n^{th} car and the $(n-1)^{\text{th}}$ car. In a subsequent paper, Herman et al. [2] have used a treatment of Laplace transform techniques on the car following model. The following differential-difference equation describes the motion of the cars in the platoon:

$$x_n''(t+T) = -\lambda [x_n'(t) - x_{n-1}'(t)]. \quad (1)$$

The constant λ is called the sensitivity constant. If the delay time T is small, as should be the case in driving, we obtain the following second order differential-difference equation:

$$x_n''(t) = -\lambda [x_n'(t) - x_{n-1}'(t)]. \quad (2)$$

From equation (2), the corresponding equation for velocities is

$$v_n'(t) = -\lambda [v_n(t) - v_{n-1}(t)]. \quad (3)$$

In section 2, we use the Laplace transforms with given initial velocities to analyze equation (3). The direct integration and differential operator techniques are used in section 3. In section 4 we use Taylor series expansion to analyze the differential difference equation (1). For a more detailed discussion of the car-following models see chapter 3 of Haberman [3]. A table of Laplace transforms is given in Abramowitz and Stegun [4].

2. The Laplace Transform

We use the notation \bar{v}_n for the Laplace transform of $v_n(t)$. Recall that the Laplace transform of derivative is given by

$$\bar{v}'_n = s\bar{v}_n - v_n(0). \quad (4)$$

Where $v_n(0)$ is the initial velocity of the n^{th} car in platoon at time t . We now take the Laplace transform on both sides of equation (3) to obtain

$$s\bar{v}_n - v_n(0) = \lambda\bar{v}_n + \lambda\bar{v}_{n-1}. \quad (5)$$

Equation (5) reduces to the following difference equation for \bar{v}_n

$$(s + \lambda)\bar{v}_n = -\lambda\bar{v}_{n-1} + v_n(0). \quad (6)$$

The solution to the above difference equation is given by Gersting [5]

$$\bar{v}_n = \frac{\lambda^n}{(s + \lambda)^n} \left[\bar{v}_0 + \sum_{i=1}^n \frac{v_i(0)}{\lambda^i} \right]. \quad (7)$$

We use a shifting theorem and take the Laplace inverse of both sides of equation (7)

$$v_n = \lambda^n e^{-\lambda t} \left[L^{-1} \left(\frac{1}{s^n} \bar{v}_0 \right) + L^{-1} \left(\frac{1}{s^n} \sum_{i=1}^n \frac{v_i(0)}{\lambda^i} \right) \right]. \quad (8)$$

The convolution theorem and the fact that $L^{-1} \left(\frac{1}{s^n} \right) = \frac{1}{(n-1)!} t^{n-1}$ yields

$$v_n = \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \left[\int_0^t (t-\tau)^{n-1} v_0(\tau) d\tau + t^{n-1} \sum_{i=1}^n \frac{v_i(0)}{\lambda^i} \right]. \quad (9)$$

We then integrate $v_n(t)$ to obtain the position function

$$x_n(t) = \int_0^t v_n(t) dt + x_n(0). \quad (10)$$

In the next section we use direct integration to analyze equation (3).

3. Direct integration

Equation (3) is a standard example of a first order linear differential equation for v_n . Let

$$v'_n + \lambda v_n = \lambda v_{n-1}. \quad (11)$$

Multiplying both sides of equation (11) by the integrating factor $e^{\lambda t}$ we obtain

$$\frac{d}{dt} (e^{\lambda t} v_n) = \lambda e^{\lambda t} v_{n-1}. \quad (12)$$

Let $u_n(t) = e^{\lambda t} v_n$ then

$$u'_n(t) = \lambda u_{n-1}. \quad (13)$$

Note that $u_n(0) = v_n(0)$. Assume that the leading car initially slows down to speed c at the same time that all other cars in the platoon are moving with initial velocities

$$v_n(0) = d, \quad n \geq 1 \quad \text{and} \quad v_0(0) = c \quad \text{where} \quad c < d.$$

By repeated application of equation (13), we obtain

$$v_n(t) = ce^{-\lambda t} \frac{(\lambda t)^n}{n!} + d \left[\sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} \right]. \quad (14)$$

Note that expansion in equation (9) will yield equation (14) if we let $v_0(0) = c$ and $v_i(0) = d$. Thus the Laplace transforms approach and the direct integration give the same result. Now we consider some limiting cases. We take the $\lim_{n \rightarrow \infty} v_n(t)$. This gives

$$\lim_{n \rightarrow \infty} v_n(t) = e^{-\lambda t} \lim_{n \rightarrow \infty} \left[-\frac{(d-c)(\lambda t)^n}{n!} + de^{\lambda t} \right]. \quad (15)$$

Utilizing

$$\lim_{n \rightarrow \infty} \frac{(\lambda t)^n}{n!} = 0 \quad (16)$$

equation (15) can be simplified to

$$\lim_{n \rightarrow \infty} v_n(t) = d. \quad (17)$$

Equation (17) shows that the cars too far from the leading car are not affected by the sudden slow down of the leading car. We also use L'Hopital's Rule to analyze the long run time behavior. This yields

$$\lim_{n \rightarrow \infty} v_n(t) = c. \quad (18)$$

Note that the limit in equation (17) is for $n \rightarrow \infty$ (last vehicles in a long platoon) where as in equation (18) we allow $t \rightarrow \infty$ (asymptotic behavior in time).

Equation (18) indicates the obvious fact that if the leading car insists on maintaining a constant slow speed of c , then eventually all the cars in the platoon have to adjust to speed $\lim_{n \rightarrow \infty} v_n(t) = c$.

The differential operator approach also gives results consistent with results shown above. By repeated application of the differential operator $D + \lambda I$ on both sides of equation (3) we obtain

$$(D + \lambda I)^n v_n = \lambda^n v_0 \quad (19)$$

The homogenous solution to the linear differential equation (19) is given by

$$v_n(t) = e^{-\lambda t} \left(\sum_{i=0}^{n-1} c_i t^i \right), \quad (20)$$

where c_i 's are constants.

In the case that v_0 is a constant, a particular solution is

$$v_n = C. \quad (21)$$

We add homogenous and a particular solution to obtain

$$v_n(t) = e^{-\lambda t} \left(\sum_{i=0}^{n-1} c_i t^i \right) + C. \quad (22)$$

Using initial condition for velocities, equation (22) is equivalent to the equation (14). A case of particular interest is when the leading car decelerates, that is, assume that the velocity of leading car is given by

$$v_0(t) = -at + b. \quad (23)$$

The operator theory approach will yield

$$v_n(t) = e^{-\lambda t} \left(\sum_{i=0}^{n-1} c_i t^i \right) - at + b + \frac{na}{\lambda}. \quad (24)$$

Taylor series

The first order Taylor series approximation is given by

$$v'_n(t+T) = v'_n(t) + T v''_n(t). \quad (25)$$

Substitute equation (25) into equation (1) and simplify to obtain the following second order differential-difference equation

$$T v''_n(t) + v'_n(t) + \lambda v_n(t) = \lambda v_{n-1}(t). \quad (26)$$

Also recall that, $L(y'') = s^2 L(y) - sy(0) - y'(0)$.

Using \bar{v}_n for the Laplace transforms of v_n and taking the inverse a solution can be obtained recursively for velocity of n^{th} car.

In summary, the Laplace transform has been used to discuss the linear car-following model. For example, assuming that the leading car has constant velocity $d < c$, we derived equation (14). Assuming that the leading car has constant deceleration we obtained equation

(24). First order Taylor series approximation was used to discuss velocity of the n^{th} car in the platoon.

Acknowledgement

The authors wish to thank Dr. Feraydune Kashefi for useful discussions related to this work.

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