LIBERTAS MATHEMATICA, vol XXVIII (2008)

Homogeneous Quadratic Dynamical Systems on $\mathbb{R}^3$ Having Derivations with Complex Eigenvalues

Ilie BURDUJAN

Abstract. The commutative binary algebras on $\mathbb{R}^3$ having a derivation with a complex eigenvalue are classified up to an isomorphism. More exactly, it is proved that there exists 20 such classes of nonnull algebras which are nonisomorphic each other. Consequently, there exist 20 classes of homogeneous quadratic (nontrivial) differential systems affinely nonequivalent each other.

Keywords: dynamical system, quadratic dynamical system, commutative binary algebra.
Mathematics subject classification (2000): 37E99, 13A99.

1 Introduction

Quadratic dynamical systems comes mainly from quadratic differential systems (briefly, QDSs). A commutative binary algebra on $\mathbb{R}^n$ is naturally associated with every homogeneous quadratic differential system (HQDS) on $\mathbb{R}^n$. This kind of associating HQDSs with commutative algebras allows us to define a 1-to-1 correspondence between the classes of affinely equivalent HQDSs on $\mathbb{R}^n$ and the classes of isomorphic commutative binary algebras on $\mathbb{R}^n$. Consequently, the classification up to an affinity of HQDSs is equivalent with the classification up to an isomorphism of corresponding commutative binary algebras. The automorphisms and derivations of any HQDS are certainly automorphisms and derivations for the corresponding binary algebra.

The QDSs on $\mathbb{R}^2$ were already classified up to an affinity according with several classifying criteria. A lot of results are obtained for QDSs defined on $\mathbb{R}^3$. The aim of this paper is to classify the HQDSs on $\mathbb{R}^3$ which admit a derivation having a complex eigenvalue. It is proved that there exist 20 classes of nonequivalent (up to an affinity) HQDSs.

2 Preliminaries

Let us consider the autonomous differential system on $\mathbb{R}^n$

$$\frac{dX}{dt} = F(X) \tag{2.1}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$ a smooth function.

An *automorphism* of (2.1) (more exactly, an automorphism of F) is an invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$F(T(X)) = T(F(X)) \tag{2.2}$$

for all $X \in \mathbb{R}^n$. The set of all automorphisms of F will be denoted by *Aut F*; it is a closed LIE subgroup of $G\ell(n, \mathbb{R})$.

A *derivation* of F is a linear transformation $D : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$DF(X) = F'(X) \cdot DX \tag{2.3}$$

for all $X \in \mathbb{R}^n$. The set of all derivations of F will be denoted by *Der F*; it is a LIE subalgebra of $g\ell(n, \mathbb{R})$. Actually, *Der F* is just the LIE algebra of *Aut F* (see [5]).

Recall that an autonomous differential system on $\mathbb{R}^n$ where F is a quadratic vector form (F is a two degree homogeneous function, i.e. $F(sX) = s^2F(X)$, for all $s \in \mathbb{R}$ and $X \in \mathbb{R}^n$) is called an *homogeneous quadratic differential system* (briefly, HQDS). In this case, the polar form G of F (i.e., the bilinear symmetric form defined by $G(X,Y) = \frac{1}{2}[F(X+Y) - F(X) - F(Y)]$ for all $X, Y \in \mathbb{R}^n$) allows us to endow $\mathbb{R}^n$ with a commutative binary operation $X \cdot Y = G(X,Y)$ for all $X, Y \in \mathbb{R}^n$; we shall denote this algebra by $A(\cdot)$. Conversely, with any real n-dimensional commutative algebra $A(\cdot)$, a HQDS having as coefficients the structure constants (in a chosen basis) of A is associated. These associations induce the existence of a 1-to-1 correspondence between the classes of affine equivalent HQDSs and the classes of isomorphic commutative algebras. This time, every automorphism (resp., derivation) of F is an automorphism (resp. a derivation) for $A(\cdot)$.

Since our goal is to classify up an affine equivalence the HQDSs on $\mathbb{R}^3$ having a special kind of derivations what is equivalent with classifying the isomorphic commutative algebras with such a kind of derivations, it is natural to give a suitable criterion for recognizing whether two algebras are or not isomorphic. The following results works as such a criterion.

Proposition 2.1 *If $T : A(\cdot) \to B(*)$ is an algebra isomorphism and $e \in A$ is a nilpotent (resp. idempotent), then $T(e)$ is also a nilpotent (resp., idempotent) of B having the same spectrum as e.*

(Recall that the *spectrum* of e is the family of all eigenvalues of the left multiplication L_e.)

3 3-Dimensional commutative algebras having a derivation with complex eigenvalues

Let $A(\cdot)$ be a commutative algebra on $\mathbb{R}^3$ having a derivation D_0 with the eigenvalues $\lambda_1 = \lambda$ and $\lambda_{2,3} = \alpha \pm i\beta$ $(\beta \neq 0)$. It must be considered the following two complementary cases:

- case $a)$ $\lambda \neq 0$,
- case $b)$ $\lambda = 0$.

Case a)

The derivation $D = \frac{1}{\lambda} D_0$ has the eigenvalues $\lambda_1 = 1$, $\lambda_{2,3} = a \pm ib$ $(b \neq 0)$ where $a = \frac{\alpha}{\lambda}$, $b = \frac{\beta}{\lambda}$. The presence of complex eigenvalues requires to use the complex extension of D to the complexification $A_{\mathbb{C}}$ of A (this extension will be also denoted by D). Since this extension of D has three different eigenvalues, there exist three linearly independent eigenvectors $e_1, e_2 \pm ie_3 \in A_{\mathbb{C}}$ such that $D(e_1) = e_1$, $D(e_2 \pm ie_3) = (a \pm ib)(e_2 \pm e_3)$. Consequently, $\mathcal{B} = (e_1, e_2, e_3)$ is a basis in $A \equiv \mathbb{R}^3$ such that

$$D(e_1) = e_1, \quad D(e_2) = ae_2 - be_3, \quad D(e_3) = be_2 + ae_3.$$

Since $D(e_1^2) = 2e_1^2$, it results $e_1^2 = 0$; similarly, the equalities

$$D(e_1 \cdot (e_2 + ie_3)) = (1 + a + ib)(e_1 \cdot (e_2 + ie_3)),$$
$$D((e_2 + ie_3)^2) = 2(a + ib)(e_2 + ie_3)^2,$$

imply the equalities $e_1 \cdot (e_2 + ie_3) = (e_2 + ie_3)^2 = 0$, i.e.,

$$e_1 \cdot e_2 = e_1 \cdot e_3 = 0,$$
$$e_2^2 = e_3^2, \quad e_2 \cdot e_3 = 0.$$

Further, if $a \neq \frac{1}{2}$, the equality $D(e_2^2) = 2ae_2^2$ gives

$$e_2^2 = e_3^2 = 0.$$

Consequently, the null algebra on $\mathbb{R}^3$ is the only algebra having a derivation with the real eigenvalue $\lambda = 1$ and a pair of complex eigenvalues $\lambda_{2,3} = a \pm ib$ with $a \neq \frac{1}{2}$.

In the case when $a = \frac{1}{2}$ then

$$e_2^2 = e_3^2 = \varepsilon e_1, \quad \varepsilon \in \mathbb{R}.$$

Consequently, in the basis $(\varepsilon e_1, e_2, e_3)$, the multiplication table of this algebra is

Table T: $e_1^2 = e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0, \;\; e_2^2 = e_3^2 = e_1.$

Derivations of $A(\cdot)$

If D denotes a derivation of $A(\cdot)$, then

$$De_i = \sum_{j=1}^{3} s_{ji} e_j, \quad i = 1, 2, 3.$$

By imposing to D to be a derivation it results:

$$\begin{aligned}
&D(e_1\cdot e_2)=D(e_1)\cdot e_2+e_1\cdot D(e_2)\Leftrightarrow s_{21}=0,\\
&D(e_1\cdot e_3)=D(e_1)\cdot e_3+e_1\cdot D(e_3)\Leftrightarrow s_{31}=0,\\
&D(e_2^2)=2e_2\cdot D(e_2)\Leftrightarrow D(e_1)=2e_2\cdot D(e_2)\Leftrightarrow s_{11}=2s_{22}\\
&D(e_2\cdot e_3)=D(e_2)\cdot e_3+e_2\cdot D(e_3)\Leftrightarrow 0=s_{32}e_1+s_{23}e_1\Leftrightarrow s_{32}+s_{23}=0,\\
&D(e_3^2)=2e_3\cdot D(e_3)\Leftrightarrow D(e_1)=2e_3\cdot D(e_3)\Leftrightarrow s_{11}=2s_{33}.
\end{aligned}$$

Consequently, the matrix of any derivation D of $A(\cdot)$ has necessarily, in basis $\mathcal{B}$, the form:

$$[D]_{\mathcal{B}}=\begin{bmatrix} s_{11} & s_{12} & s_{13}\\ 0 & \frac{1}{2}s_{11} & s_{23}\\ 0 & -s_{23} & \frac{1}{2}s_{11}\end{bmatrix},$$

where $s_{11},s_{12},s_{13},s_{23}\in\mathbb{R}$; conversely, every endomorphism of A having such a matrix in basis $\mathcal{B}$ is a derivation of $A(\cdot)$. Moreover, it results

$$[D]_{\mathcal{B}}=s_{11}[D_1]_{\mathcal{B}}+s_{23}[D_2]_{\mathcal{B}}+s_{12}[D_3]_{\mathcal{B}}+s_{13}[D_4]_{\mathcal{B}}$$

where

$$[D_1]_{\mathcal{B}}=\frac{1}{2}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad [D_2]_{\mathcal{B}}=\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix},$$
$$[D_3]_{\mathcal{B}}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\quad [D_4]_{\mathcal{B}}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}.$$

A straightforward checking shows us that the endomorphisms D_1, D_2, D_3 şi D_4 represented in basis $\mathcal{B}$ by matrices $[D_1]_{\mathcal{B}}$, $[D_2]_{\mathcal{B}}$, $[D_3]_{\mathcal{B}}$ and, respectively, $[D_4]_{\mathcal{B}}$ are derivations of $A(\cdot)$. Moreover, *Der* A has the structure constants defined by the following equalities

$$\begin{aligned}
&[D_1,D_2]=0, && [D_1,D_3]=\frac{1}{2}D_3, && [D_1,D_4]=\frac{1}{2}D_4\\
&[D_2,D_3]=-D_4, && [D_2,D_4]=D_3, && [D_3,D_4]=0.
\end{aligned}$$

It results that *Der* $A=\mathbb{R}D_1\oplus\mathbb{R}D_2\oplus\mathbb{R}D_3\oplus\mathbb{R}D_4$. Consequently, there exists a 4-parametric group of automorphisms for $A(\cdot)$.

Remark. As D_2 is necessarily a derivation of $A(\cdot)$ it results that $A(\cdot)$ will be also present on the list of algebras appearing in the Case b).

Automorphisms of $A(\cdot)$

Taking in account that A has the derivations D_1, D_2, D_3 and D_4, it results that $\{e^{tD_1}|t\in\mathbb{R}\}$, $\{e^{\tau D_2}|\tau\in\mathbb{R}\}$, $\{e^{\theta D_3}|\theta\in\mathbb{R}\}$ and $\{e^{\vartheta D_4}|\vartheta\in\mathbb{R}\}$ are uniparametric subgroups of automorphisms of this algebra. In basis $\mathcal{B}$ these uniparametric subgroups are realized as the following matrix subgroups

$$[e^{tD_1}]_B = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{\frac{t}{2}} & 0 \\ 0 & 0 & e^{\frac{t}{2}} \end{bmatrix}, \quad [e^{\tau D_2}]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau \\ 0 & -\sin\tau & \cos\tau \end{bmatrix},$$
$$[e^{\theta D_3}]_B = \begin{bmatrix} 1 & \theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [e^{\vartheta D_4}]_B = \begin{bmatrix} 1 & 0 & \vartheta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $t, \tau, \theta, \vartheta \in \mathbb{R}$. Indeed, for example,

$$[e^{\tau D_2}]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\tau}{1!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} + \frac{\tau^2}{2!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} +$$
$$+\frac{\tau^3}{3!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + ... = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau \\ 0 & -\sin\tau & \cos\tau \end{bmatrix}, \quad \forall \tau \in \mathbb{R}.$$

In order to establish whether exist or not other automorphisms of this algebra we proceed by a direct checking.

Any automorphism $T : A(\cdot) \to A(\cdot)$ has the form:

$$\begin{aligned} T(e_1) &= a_{11}e_1 + a_{21}e_2 + a_{31}e_3 \\ T(e_2) &= a_{12}e_1 + a_{22}e_2 + a_{32}e_3 \\ T(e_3) &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 \end{aligned}$$

Since every automorphism carries any special element (such as, annulator, nilpotent, idempotent) on a special element of the same kind, it results, necessarily, that the following condition is satisfied

$$T(e_1) = a_{11}e_1,$$

with $a_{11} \neq 0$. Further, it results

$$\begin{aligned} &T(e_2^2) = (T(e_2))^2 \Leftrightarrow T(e_1) = (a_{12}e_1 + a_{22}e_2 + a_{32}e_3)^2 \Leftrightarrow \\ &\Leftrightarrow a_{11}e_1 = [(a_{22})^2 + (a_{32})^2]e_1 \Leftrightarrow a_{11} = (a_{22})^2 + (a_{32})^2, \\ &T(e_2e_3) = T(e_2)\cdot T(e_3) \Leftrightarrow 0 = (a_{22}a_{23} + a_{32}a_{33})e_1 \Leftrightarrow 0 = a_{22}a_{23} + a_{32}a_{33} \\ &T(e_3^2) = (T(e_3))^2 \Leftrightarrow T(e_1) = (a_{13}e_1 + a_{23}e_2 + a_{33}e_3)^2 \Leftrightarrow \\ &\Leftrightarrow a_{11}e_1 = [(a_{23})^2 + (a_{33})^2]e_1 \Leftrightarrow a_{11} = (a_{23})^2 + (a_{33})^2. \end{aligned}$$

Therefore, the matrix of every automorphism of the algebra $A(\cdot)$ has necessarily the form:

$$[T]_B = \begin{bmatrix} \rho^2 & \beta & \gamma \\ 0 & \rho\cos\alpha & \rho\sin\alpha \\ 0 & -\rho\sin\alpha & \rho\cos\alpha \end{bmatrix}$$

with $\rho, \alpha, \beta, \gamma \in \mathbb{R}$ şi $\rho > 0$; conversely, every endomorphism of A, having such a matrix in basis B is an automorphism of $A(\cdot)$.

These matrices are consisting in a 4-parametric group. Indeed, if it is used the notation

$$[T]_{\mathcal{B}} = T(\rho, \alpha, \beta, \gamma),$$

with $\rho, \alpha, \beta, \gamma \in \mathbb{R}$ then $T(\rho, \alpha, \beta, \gamma)T(\varrho, \lambda, \mu, \eta) = T(\rho\varrho, \alpha+\lambda, \rho^2\mu+\varrho\beta cos\ \lambda - \varrho\gamma sin\ \lambda, \rho^2\eta + \varrho\beta sin\ \lambda + \varrho\gamma cos\ \lambda)$, for all $\rho, \alpha, \beta, \gamma, \varrho, \lambda, \mu, \eta \in \mathbb{R}$ with $\rho > 0$, $\varrho > 0$. This matrix group is just the 4-parametric group associated with all derivations D of A, which was being exhibited at the beginning of this section.

Therefore, **the GALOIS group of $A(\cdot)$ is isomorphic with the before defined matrix group $\{T(\rho, \alpha, \beta, \gamma) | \rho, \alpha, \beta, \gamma \in \mathbb{R}, \rho > 0\}$.**

The equations of any automorphism $T(\rho, \alpha, \beta, \gamma)$ are

$$\begin{cases} \overline{x}^1 = & \rho^2\, x^1 + \beta\, x^2 + \gamma\, x^3, \\ \overline{x}^2 = & \rho cos\ \alpha\, x^2 + \rho sin\ \alpha\, x^3, \\ \overline{x}^3 = & -\rho sin\ \alpha\, x^2 + \rho cos\ \alpha\, x^3. \end{cases}$$

Then, the infinitesimal operators of $Aut\ A$ are

$$\begin{aligned} X_1 &= 2x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}, \\ X_2 &= x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3}, \\ X_3 &= x^2\frac{\partial}{\partial x^1}, \quad X_4 = x^3\frac{\partial}{\partial x^1}. \end{aligned}$$

Obviously, they generate the LIE algebra of vector fields having the following structure equations

$$\begin{array}{lll} [X_1, X_2] = 0, & [X_1, X_3] = -X_3, & [X_1, X_4] = -X_4 \\ [X_2, X_3] = X_4, & [X_2, X_4] = -X_3, & [X_3, X_4] = 0. \end{array}$$

This LIE algebra is isomorphic with $Der\ A$.

Remark. By identifying the automorphism with a change of basis, it results that the corresponding change of coordinates is a contravariant transformation, i.e., it has the equations

$$\begin{cases} \overline{x}^1 = & \frac{1}{\rho^2}\, x^1 + \frac{1}{\rho^3}[(-\beta cos\ \alpha\ + \gamma sin\ \alpha)x^2 + (\beta sin\ \alpha - \gamma cos\ \alpha)\, x^3], \\ \overline{x}^2 = & \frac{1}{\rho}(cos\ \alpha\, x^2 - sin\ \alpha\, x^3), \\ \overline{x}^3 = & -\frac{1}{\rho^2}(sin\ \alpha\, x^2 + \rho cos\ \alpha\, x^3). \end{cases}$$

The corresponding infinitesimal operators are $\{-X_1, -X_2, -X_3\}$. In basis $\{Y_1 = -\frac{1}{2}X_1$, $Y_2 = -X_2$, $Y_3 = -X_3\}$ the structure constants of the LIE algebra spanned by these vector fields are identical with those of $Der\ A$ in basis $\{D_1, D_2, D_3, D_4\}$.

Case b)

Let D_0 be a derivation with $\lambda_1 = 0, \lambda_{2,3} = \alpha + i\beta$ $(\beta \neq 0)$.

We have to consider the cases:

- case b_1: $\alpha \neq 0$,
- case b_2: $\alpha = 0$.

Case b_1

The derivation $D = \frac{1}{\beta} D_0$ has the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = a \pm i$ with $a = \frac{\alpha}{\beta} \neq 0$ and there exists a basis $B = (e_1, e_2, e_3)$ such that

$$D(e_1) = 0, \quad D(e_2) = ae_2 - e_3, \quad D(e_3) = e_2 + ae_3.$$

Then the equalities

$$\begin{aligned} &D(e_1^2) = 0, \\ &D(e_1 \cdot (e_2 + ie_3)) = (a+i)(e_1 \cdot (e_2 + ie_3)), \\ &D((e_2 + ie_3)^2) = 2(a+i)(e_2 + ie_3)^2, \end{aligned}$$

are equivalent with

$$\begin{aligned} &e_1^2 = \varepsilon e_1, \\ &e_1 \cdot (e_2 + ie_3) = (\kappa + i\omega)(e_2 + ie_3), \\ &(e_2 + ie_3)^2 = 0. \end{aligned}$$

It results

$$\begin{aligned} &e_1 \cdot e_2 = \kappa e_2 - \omega e_3, \\ &e_1 \cdot e_3 = \omega e_2 + \kappa e_3, \\ &e_2^2 = e_3^2, \quad e_2 e_3 = 0, \end{aligned}$$

with $\kappa, \omega \in \mathbb{R}$. Since $D(e_2^2) = 2ae_2^2$ and $a \neq 0$ it results $e_2^2 = e_3^2 = 0$. Consequently, the multiplication table of the algebra, in basis B, has the form

Table T' $\quad e_1^2 = \varepsilon e_1, \ \ e_1 \cdot e_2 = \kappa e_2 - \omega e_3, \ \ e_1 \cdot e_3 = \omega e_2 + \kappa e_3,$
$\qquad e_2^2 = e_3^2 = e_2 \cdot e_3 = 0,$

with $\kappa, \omega \in \mathbb{R}$. By passing to the basis $\left(\frac{1}{\varepsilon} e_1, e_2, e_3\right)$ and putting κ, ω respectively instead of $\frac{\kappa}{\varepsilon}, \frac{\omega}{\varepsilon}$, the following multiplication table is obtained

Table T" $\quad e_1^2 = e_1, \ \ e_1 \cdot e_2 = \kappa e_2 - \omega e_3, \ \ e_1 \cdot e_3 = \omega e_2 + \kappa e_3,$
$\qquad e_2^2 = e_3^2 = e_2 \cdot e_3 = 0,$

with $\kappa, \omega \in \mathbb{R}$, $\omega \neq 0$. We shall denote with $A(\kappa, \omega)$ the algebra with the multiplication table **T"**. Using the basis $(e_1, e_2, -e_3)$ it results that $A(\kappa, \omega)$ is isomorphic with $A(\kappa, -\omega)$. It results that only algebras $A(\kappa, \omega)$ with $\kappa, \omega \in \mathbb{R}$ and $\omega > 0$ will be of interest.

For such algebras $Ann\ A = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\}$, $\mathcal{I}(A) = \{e_1\}$ and A is a vector direct sum of a subalgebra and an ideal, namely $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$ (i.e., this is a WEDDERBURN-ARTIN decomposition for A).

Isomorphisms of algebras $A(\kappa, \omega)$.

Problem: *Can be isomorphic two algebras $A(\kappa_1, \omega_1)$ and $A(\kappa_2, \omega_2)$ with $\kappa_1, \omega_1, \kappa_2, \omega_2 \in \mathbb{R}$ and $\omega_1 > 0$, $\omega_2 > 0$?*

Firstly, we recall that any algebra of type **T"** has $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\}$ and a unique idempotent.

Let us consider the algebras $A(\kappa_1, \omega_1)$ and $A(\kappa_2, \omega_2)$, with the multiplication tables

Table T$_1$" $\quad e_1^2 = e_1, \quad e_1 \cdot e_2 = \kappa_1 e_2 - \omega_1 e_3, \quad e_1 \cdot e_3 = \omega_1 e_2 + \kappa_1 e_3,$
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0,$

and, respectively,

Table T$_2$" $\quad f_1^2 = f_1, \quad f_1 \cdot f_2 = \kappa_2 f_2 - \omega_2 f_3, \quad f_1 \cdot f_3 = \omega_2 f_2 + \kappa_2 f_3,$
$f_2^2 = f_3^2 = f_2 \cdot f_3 = 0,$

These algebras have a unique nonzero nilpotent ideal $J = Span_{\mathbb{R}}\{e_2, e_3\}$ and respectively $J' = Span_{\mathbb{R}}\{f_2, f_3\}$; further, they have the vector sum decompositions

$$A = \mathbb{R}e_1 \oplus J, \quad A = \mathbb{R}f_1 \oplus J'.$$

An isomorphism $T : A(\kappa_1, \omega_1) \to A(\kappa_2, \omega_2)$ must respect the before presented vector direct sum decompositions, so that it has the form

$$\begin{aligned} T(e_1) &= f_1, \\ T(e_2) &= s_{22} f_2 + s_{32} f_3, \\ T(e_3) &= s_{23} f_2 + s_{33} f_3 \end{aligned}$$

with $s_{22}s_{33} - s_{23}s_{32} \neq 0$. The following equalities hold:

$$\begin{aligned} &T(e_1 \cdot e_2) = \kappa_1 T(e_2) - \omega_1 T(e_3) = (\kappa_1 s_{22} - \omega_1 s_{23}) f_2 + (\kappa_1 s_{32} - \omega_1 s_{33}) f_3 \\ &T(e_1) \cdot T(e_2) = (\kappa_2 s_{22} + \omega_2 s_{32}) f_2 + (-\omega_2 s_{22} + \kappa_2 s_{32}) f_3, \\ &T(e_1 \cdot e_3) = \omega_1 T(e_2) + \kappa_1 T(e_3) = (\omega_1 s_{22} + \kappa_1 s_{23}) f_2 + (\omega_1 s_{32} + \kappa_1 s_{33}) f_3 \\ &T(e_1) \cdot T(e_3) = s_{23}(\kappa_2 f_2 - \omega_2 f_3) + s_{33}(\omega_2 f_2 + \kappa_2 f_3) = \\ &= (\kappa_2 s_{23} + \omega_2 s_{33}) f_2 + (-\omega_2 s_{23} + \kappa_2 s_{33}) f_3. \end{aligned}$$

Consequently, we get the equalities:

$$\begin{cases} (\kappa_1 - \kappa_2) s_{22} - \omega_1 s_{23} - \omega_2 s_{32} = 0, \\ \omega_2 s_{22} + (\kappa_1 - \kappa_2) s_{32} - \omega_1 s_{33} = 0, \\ \omega_1 s_{22} + (\kappa_1 - \kappa_2) s_{23} - \omega_2 s_{33} = 0, \\ \omega_2 s_{23} + \omega_1 s_{32} + (\kappa_1 - \kappa_2) s_{33} = 0. \end{cases}$$

Taking into account that $s_{22}, s_{23}, s_{32}, s_{33}$ cannot vanish simultaneously, these relations imply:

$$\Delta = det \begin{bmatrix} \kappa_1 - \kappa_2 & -\omega_1 & -\omega_2 & 0 \\ \omega_2 & \kappa_1 - \kappa_2 & 0 & -\omega_1 \\ \omega_1 & 0 & \kappa_1 - \kappa_2 & -\omega_2 \\ 0 & \omega_2 & \omega_1 & \kappa_1 - \kappa_2 \end{bmatrix} = 0.$$

But $\omega_1 > 0$, $\omega_2 > 0$ and

$$\Delta = -(\kappa_1 - \kappa_2)^2[(\kappa_1 - \kappa_2)^2 + 2(\omega_1)^2 + 2(\omega_2)^2] - [(\omega_1)^2 - (\omega_2)^2]^2 = 0$$

imply

$$\kappa_1 = \kappa_2, \quad \omega_1 = \omega_2.$$

Therefore, we have proved the following result:

Proposition 3.1 *The algebras $A(\kappa_1, \omega_1)$ and $A(\kappa_2, \omega_2)$ with $\omega_1 > 0$, $\omega_2 > 0$ are isomorphic if and only if $\kappa_1 = \kappa_2$ şi $\omega_1 = \omega_2$.*

Derivations of algebra $A(\kappa, \omega)$

If D denotes an arbitrary derivation of A, then

$$De_i = \sum_{j=1}^{3} s_{ji} e_j, \quad i = 1, 2, 3.$$

By imposing D be a derivation and taking in account that $\omega > 0$ it results:

$$\begin{aligned}
&D(e_1^2) = 2e_1 \cdot D(e_1) \Leftrightarrow D(e_1) = 2e_1 \cdot D(e_1) \Leftrightarrow\\
&\Leftrightarrow s_{11} = 0, s_{21} = 2\kappa s_{21} + 2\omega s_{31}, s_{31} = -2\omega s_{21} + 2\kappa s_{31} \Leftrightarrow\\
&\Leftrightarrow s_{11} = s_{21} = s_{31} = 0 \Leftrightarrow D(e_1) = 0,\\
&D(e_1 \cdot e_2) = \kappa De_2 - \omega De_3 = e_1 \cdot D(e_2) \Leftrightarrow\\
&\Leftrightarrow (\kappa s_{12} - \omega s_{13})e_1 + (\kappa s_{22} - \omega s_{23})e_2 + (\kappa s_{32} - \omega s_{33})e_3 =\\
&= s_{12}e_1 + s_{22}(\kappa e_2 - \omega e_3) + s_{32}(\omega e_2 + \kappa e_3) \Leftrightarrow\\
&\Leftrightarrow \kappa s_{12} - \omega s_{13} = s_{12},\ \kappa s_{22} - \omega s_{23} = \kappa s_{22} + \omega s_{32},\ \kappa s_{32} - \omega s_{33} =\\
&= -\omega s_{22} + \kappa s_{32} \Leftrightarrow (\kappa - 1)s_{12} - \omega s_{13} = 0,\ s_{23} = -s_{32},\ s_{22} = s_{33},\\
&D(e_1 \cdot e_3) = \omega D(e_2) + \kappa D(e_3) = e_1 \cdot D(e_3) \Leftrightarrow\\
&\Leftrightarrow (\omega s_{12} + \kappa s_{13})e_1 + (\omega s_{22} + \kappa s_{23})e_2 + (\omega s_{32} + \kappa s_{33})e_3 =\\
&= s_{13}e_1 + s_{23}(\kappa e_2 - \omega e_3) + s_{33}(\omega e_2 + \kappa e_3) \Leftrightarrow\\
&\Leftrightarrow \omega s_{12} + \kappa s_{13} = s_{13},\ \omega s_{22} + \kappa s_{23} = \kappa s_{23} + \omega s_{33},\\
&\omega s_{32} + \kappa s_{33} = -\omega s_{23} + \kappa s_{33} \Leftrightarrow \omega s_{12} + (\kappa - 1)s_{13} = 0.
\end{aligned}$$

The relations

$$\begin{cases} (\kappa - 1)s_{12} - \omega s_{13} = 0,\\ \omega s_{12} + (\kappa - 1)s_{13} = 0, \end{cases}$$

imply

$$s_{12} = s_{13} = 0.$$

The before obtained necessary conditions assure that all conditions of being a derivation are fulfilled for D. Consequently, any derivation D of $A(\kappa, \omega)$ has, in basis B, the matrix:

$$[D]_B = \begin{bmatrix} 0 & 0 & 0\\ 0 & s_{22} & s_{23}\\ 0 & -s_{23} & s_{22} \end{bmatrix}.$$

It results that

$$[D]_B = s_{22}[D_1]_B + s_{23}[D_2]_B$$

where

$$[D_1]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [D_2]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

It is easy to prove that the endomorphisms D_1 şi D_2 represented in basis B by the matrices $[D_1]_B$ and $[D_2]_B$ respectively, are derivations of $A(\kappa, \omega)$ (D_2 is just the former derivation of A).

It results that $Der\ A = \mathbb{R}D_1 \oplus \mathbb{R}D_2$ **is an Abelian LIE algebra. This means that there exists a biparametric commutative group of automorphisms of** $A(\kappa, \omega)$**.**

Remark. As D_2 is necessarily a derivation of $A(\kappa, \omega)$, this algebra will appear also on the list of algebras for the Case b_2.

Automorphisms of algebra $A(\kappa, \omega)$

As the algebra A has the derivations D_1 and D_2 it results that $\{e^{tD_1} | t \in \mathbb{R}\}$ and $\{e^{\tau D_2} | \tau \in \mathbb{R}\}$ are uniparametric groups of automorphisms of it. In the basis B these uniparametric subgroups are realized as following matrix subgroups

$$[e^{tD_1}]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}, \quad [e^{tD_2}]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau \\ 0 & -\sin\tau & \cos\tau \end{bmatrix},$$

with $t, \tau \in \mathbb{R}$.

In order to establish if there exists or not other new automorphisms of the algebra we proceed by a straightforward checking.

Any automorphism $T : A(\kappa, \omega) \to A(\kappa, \omega)$ has the form:

$$T(e_1) = e_1,$$
$$T(e_2) = a_{22}e_2 + a_{32}e_3$$
$$T(e_3) = a_{23}e_2 + a_{33}e_3$$

with $a_{22}a_{33} - a_{23}a_{32} \neq 0$. The equalities

$$T(e_1e_2) = T(e_1) \cdot T(e_2), \Leftrightarrow \kappa T(e_2) - \omega T(e_3) = e_1(a_{22}e_2 + a_{32}e_3) \Leftrightarrow$$
$$\Leftrightarrow (\kappa a_{22} - \omega a_{23})e_2 + (\kappa a_{32} - \omega a_{33})e_3 = a_{22}(\kappa e_2 - \omega e_3) + a_{32}(\omega e_2 + \kappa e_3), \Leftrightarrow$$
$$\Leftrightarrow \kappa a_{22} - \omega a_{23} = \kappa a_{22} + \omega a_{32}, \quad \kappa a_{32} - \omega a_{33} = -\omega a_{22} + \kappa a_{32},$$

imply

$$a_{32} = -a_{23}, \quad a_{22} = a_{33}.$$

Consequently, the matrix of any automorphism of the algebra $A(\kappa, \omega)$ has the form:

$$[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & -a_{23} & a_{22} \end{bmatrix}$$

with $a_{22}, a_{23} \in \mathbb{R}$ and $(a_{22})^2 + (a_{23})^2 \neq 0$. A direct computation shows that any endomorphism of A having such a matrix is an automorphism for $A(\cdot)$.

All matrices of automorphisms of $A(\cdot)$ form a biparametric group. Indeed, if we shall use the notation

$$T(\alpha,\beta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \alpha & \beta\\ 0 & -\beta & \alpha\end{bmatrix},$$

with $\alpha,\beta\in\mathbb{R}$ then $T(\alpha,\beta)T(\gamma,\delta)=T(\alpha\gamma-\beta\delta,\alpha\delta+\beta\gamma)\ \forall\alpha,\beta\in\mathbb{R}$. This group is just the biparametric (commutative) group associated with the LIE algebra *Der A*.

Consequently, **the GALOIS group of the algebra $A(\kappa,\omega)$ is isomorphic with the before defined matrix group $\{T(\alpha,\beta)|\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2\neq 0\}$.**

The equations of the automorphism $T(\alpha,\beta)$ are

$$\begin{cases}\overline{x}^1= x^1,\\ \overline{x}^2=\alpha x^2-\beta x^3,\\ \overline{x}^3=\beta x^2+\alpha x^3.\end{cases}$$

Then, the associated infinitesimal operators of this GALOIS group are

$$X_1= x^2\frac{\partial}{\partial x^2}+x^3\frac{\partial}{\partial x^3},$$
$$X_2=-x^3\frac{\partial}{\partial x^2}+x^2\frac{\partial}{\partial x^3}.$$

It can be proved that $[X_1,X_2]=0$, what is in accord with the fact that the LIE algebra *Der A* is Abelian.

The homogeneous quadratic differential system associated with algebra $A(\kappa,\omega)$

By using the usual procedure the following HQDS is associated with the algebra $A(\kappa,\omega)$:

$$\begin{cases}\dot{x}^1= (x^1)^2\\ \dot{x}^2= 2\kappa x^1x^2+2\omega x^1x^3\\ \dot{x}^3=-2\omega x^1x^2+2\kappa x^1x^3.\end{cases}$$

This system can be solved. Indeed, first equation has the general solution $x^1(t)=\dfrac{a}{1-at}$. Using this $x^1(t)$ in the second and the third equation one gets a linear system of differential equations for which it can be determined two independent prime integrals, namely:

$$\begin{cases}F_1(x^1,x^2,x^3)=\dfrac{(x^2)^2+(x^3)^2}{(x^1)^{4\kappa}},\\ F_2(x^1,x^2,x^3)=arctg\,\dfrac{x^2}{x^3}-(x^1)^{2\omega}.\end{cases}$$

This assertion is obtained by constructing the entities $2x^2\dot{x}^2+2x^3\dot{x}^3$, $x^3\dot{x}^2-x^2\dot{x}^3$ and taking in account that on every orbit $x^1=\dfrac{\dot{x}^1}{x^1}$.

The vector field associated with this system is

$$Y = (x^1)^2\frac{\partial}{\partial x^1} + 2x^1(\kappa x^2 + \omega x^3)\frac{\partial}{\partial x^2} - 2x^1(\omega x^2 - \kappa x^3)\frac{\partial}{\partial x^3}.$$

By a straightforward computation the following equalities are checked:

$$[X_1, Y] = [X_2, Y] = 0;$$

they assure that X_1, X_2 belong to the centralizer of Y in the Lie algebra of all polynomial vector fields in $\mathbb{R}^3$.

Case $\mathbf{b_2}$

The algebra A has the derivation $D = \frac{1}{\beta}D_0$ with the eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm i$. In this case, there exists a basis $\mathcal{B} = (e_1, e_2, e_3)$ such that

$$De_1 = 0, \quad De_2 = -e_3, \quad De_3 = e_2.$$

Consequently, the multiplication table of the algebra A in basis $\mathcal{B}$ has the form:

$$\text{Table } 1^\circ \quad \begin{array}{l} e_1^2 = ae_1, \;\; e_1 \cdot e_2 = be_2 + ce_3, \;\; e_1 \cdot e_3 = -ce_2 + be_3, \\ e_2^2 = e_3^2 = \varepsilon e_1, \;\; e_2 \cdot e_3 = 0, \end{array}$$

with $a, b, c, \varepsilon \in \mathbb{R}$.

In order to emphasize the dependence of parameters, we shall denote by $A(a, b, c, \varepsilon)$ the algebra defined by the Table 1°. The algebra $A(a, b, c, \varepsilon)$ has the subalgebra $\mathbb{R}e_1$. If $\varepsilon \neq 0$ and $b^2 + c^2 \neq 0$, then $A^2 = A$ and the following problem arises: establish whether the algebra A is or not a simple one.

Proposition 3.2 *The algebra $A(a, b, c, \varepsilon)$ with $\varepsilon \neq 0$ and $b^2 + c^2 \neq 0$ is a simple algebra.*

Proof. Let $\mathcal{I} \subset A$ be an ideal and $v = xe_1 + ye_2 + ze_3 \in \mathcal{I}$, $v \neq 0$; then the elements

$$\begin{aligned} v \cdot e_1 &= axe_1 + (by - cz)e_2 + (cy + bz)e_3, \\ v \cdot e_2 &= \varepsilon ye_1 + bxe_2 + cxe_3, \\ v \cdot e_3 &= \varepsilon ze_1 - cxe_2 + bxe_3, \end{aligned}$$

must belong to $\mathcal{I}$. Consequently, if

$$\Delta = x(b^2 + c^2)[ax^2 - \varepsilon(y^2 + z^2)]$$

is nonzero then $v \cdot e_1$, $v \cdot e_2$ şi $v \cdot e_3$ are linearly independent and, necessarily, $\mathcal{I} \equiv A$. If $x = 0$, then the equalities $v \cdot e_2 = \varepsilon ye_1$, $v \cdot e_3 = \varepsilon ze_1$ and $v \neq 0$ imply $e_1, e_1 \cdot e_2, e_1 \cdot e_3 \in \mathcal{I}$, i.e. $\mathcal{I} \equiv A$. In the case when $x \neq 0$ and $ax^2 - \varepsilon(y^2 + z^2) = 0$, the linear independence of vectors $v \cdot e_2$, $v \cdot e_3$ and $(v \cdot e_2) \cdot e_3 = \varepsilon cxe_1 + \varepsilon y(-ce_2 + be_3)(\in \mathcal{I})$ is equivalent with the nonvanishing of the determinant

$$\Delta_1 = \varepsilon x(b^2 + c^2)[cx^2 - \varepsilon yz];$$

similarly, the linear independence of vectors $v \cdot e_2$, $v \cdot e_3$ and $(v \cdot e_3) \cdot e_2 = -\varepsilon cxe_1 + \varepsilon z(be_2 + ce_3)(\in \mathcal{I})$ is equivalent with the nonvanishing of the determinant

$$\Delta_2 = -\varepsilon x(b^2 + c^2)[cx^2 + \varepsilon yz].$$

If $\Delta_1^2 + \Delta_2^2 \neq 0$ then $\mathcal{I} \equiv A$; if $\Delta_1^2 + \Delta_2^2 = 0$, then $cx^2 = \varepsilon yz = 0$ imply necessarily $c = 0$ and $b \neq 0$. Thus, it must be considered the complementary cases $y = 0$ and, respectively, $y \neq 0 (\Rightarrow z = 0)$. When $y = 0$ (necessarily, $c = 0$ and $b \neq 0$), then $e_2 = \frac{1}{bx} v \cdot e_2 \in \mathcal{I}, e_1 \cdot e_2 \in \mathcal{I}$ and $e_1 = \frac{1}{\varepsilon bx}(v \cdot e_2) \cdot e_2 \in \mathcal{I}$, $e_1 \cdot e_3 \in \mathcal{I}$, i.e. $\mathcal{I} \equiv A$. In its turn, $y \neq 0$ implies $z = 0$, i.e. $v = xe_1 + ye_2$. Then $e_3 = \frac{1}{bx} v \cdot e_3 \in \mathcal{I}, e_1 \cdot e_3 \in \mathcal{I}$ and $e_1 = \frac{1}{\varepsilon bx}(v \cdot e_3) \cdot e_3 \in \mathcal{I}$, $e_1 \cdot e_3 \in \mathcal{I}$, i.e. $\mathcal{I} \equiv A$.

The following assertions can be easily checked:

• if $\varepsilon = 0$ and $b^2 + c^2 \neq 0$, then A is the direct vector sum of the subalgebra $\mathbb{R}e_1$ with the ideal $Span_{\mathbb{R}}\{e_2, e_3\}$, i.e. $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$ is a WEDDERBURN-ARTIN decomposition of A,

• if $b = c = 0$ and $\varepsilon \neq 0$ then $A^2 = \mathbb{R}e_1$ is a nontrivial ideal,

• if $b = c = \varepsilon = 0$, then A is the direct sum of the ideals $\mathbb{R}e_1$ and $Span_{\mathbb{R}}\{e_2, e_3\}$, i.e. $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$.

It must to note that if $a = 0$ then there exist nilpotents which are collinear with e_1, while if $a \neq 0$ there exists an idempotent collinear with e_1. So, the vanishing/nonvanishing of a is reflected in the structure of the algebra. Similarly it can be remarked that the vanishing/nonvanishing of anyone of the parameters b, c or ε can be indentified with the existence of a specific propriety of the algebra which is independent of the used basis, i.e. every such a vanishing/nonvanishing have an invariant character. That is why, it is natural to consider the following cases:

1°	$a = b = c = \varepsilon = 0$,	9°	$a \neq 0, b = c = \varepsilon = 0$,
2°	$a = b = c = 0, \varepsilon \neq 0$,	10°	$a \neq 0, b = c = 0, \varepsilon \neq 0$,
3°	$a = b = 0, c \neq 0, \varepsilon = 0$,	11°	$a \neq 0, b = 0, c \neq 0, \varepsilon = 0$,
4°	$a = b = 0, c \neq 0, \varepsilon \neq 0$,	12°	$a \neq 0, b = 0, c \neq 0, \varepsilon \neq 0$,
5°	$a = 0, b \neq 0, c = \varepsilon = 0$,	13°	$a \neq 0, b \neq 0, c = \varepsilon = 0$,
6°	$a = 0, b \neq 0, c = 0, \varepsilon \neq 0$,	14°	$a \neq 0, b \neq 0, c = 0, \varepsilon \neq 0$,
7°	$a = 0, b \neq 0, c \neq 0, \varepsilon = 0$,	15°	$a \neq 0, b \neq 0, c \neq 0, \varepsilon = 0$,
8°	$a = 0, b \neq 0, c \neq 0, \varepsilon \neq 0$	16°	$a \neq 0, b \neq 0, c \neq 0, \varepsilon \neq 0$.

Case 1°

The algebra $A(0, 0, 0, 0)$ is just the null algebra on A.

Case 2°

Algebra $A(0,0,0,\varepsilon)$ with $\varepsilon \neq 0$ is isomorphic with $A(0,0,0,1)$. Indeed, in the basis $(\varepsilon e_1, e_2, e_3)$ the multiplication table becomes:

Table T2 $\quad \begin{array}{l} e_1^2 = e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0, \\ e_2^2 = e_3^2 = e_1. \end{array}$

Obviously, $A(0,0,0,1)$ is coincident with the algebra defined by Table T in Case a).

Algebra $A(0,0,0,1)$ has the properties:

- $Ann\ A = \mathbb{R}e_1, \quad \mathcal{N}(A) = \mathbb{R}e_1, \quad \mathcal{I}(A) = \emptyset$,
- it is not a simple algebra ($A^2 = \mathbb{R}e_1$).

Case 3°

The algebra $A(0,0,c,0)$ with $c \neq 0$ is isomorphic with $A(0,0,1,0)$. Indeed, in the basis $(\frac{1}{c}e_1, e_2, e_3)$ the multiplication table becomes:

Table T3 $\quad \begin{array}{l} e_1 \cdot e_2 = e_3, \ \ e_1 \cdot e_3 = -e_2, \\ e_1^2 = e_2 \cdot e_3 = e_2^2 = e_3^2 = 0. \end{array}$

Algebra $A(0,0,1,0)$ has the properties:

- $Ann\ A = \{0\}, \quad \mathcal{N}(A) = \mathbb{R}e_1 \cup Span_{\mathbb{R}}\{e_2, e_3\}, \quad \mathcal{I}(A) = \emptyset$,
- it is not a simple algebra ($Span_{\mathbb{R}}\{e_2, e_3\}$ is an ideal).

Case 4°

The algebra $A(0,0,c,\varepsilon)$ with $c\varepsilon \neq 0$ is isomorphic with $A(0,0,1,\pm 1)$, depending on either $c\varepsilon$ is positive or negative. Indeed, in the basis (e_1, e_2, e_3) the multiplication table is:

Table T4 $\quad \begin{array}{ll} e_1 \cdot e_2 = ce_3, & e_1 \cdot e_3 = -ce_2, \\ e_1^2 = e_2 \cdot e_3 = 0, & e_2^2 = e_3^2 = \varepsilon e_1. \end{array}$

Case $c\varepsilon > 0$. In the basis $\left(\frac{1}{c}e_1, \frac{1}{\sqrt{c\varepsilon}}e_2, \frac{1}{\sqrt{c\varepsilon}}e_3\right)$ the multiplication table becomes the table of algebra $A(0,0,1,1)$, namely

Table T4$_1$ $\quad \begin{array}{ll} e_1 \cdot e_2 = e_3, & e_1 \cdot e_3 = -e_2, \\ e_1^2 = e_2 \cdot e_3 = 0, & e_2^2 = e_3^2 = e_1. \end{array}$

Case $c\varepsilon < 0$. In the basis $\left(\frac{1}{c}e_1, \frac{1}{\sqrt{-c\varepsilon}}e_2, \frac{1}{\sqrt{-c\varepsilon}}e_3\right)$ the multiplication table becomes the table of the algebra $A(0,0,1,-1)$, namely

Table T4$_2$ $\quad \begin{array}{ll} e_1 \cdot e_2 = e_3, & e_1 \cdot e_3 = -e_2, \\ e_1^2 = e_2 \cdot e_3 = 0, & e_2^2 = e_3^2 = -e_1. \end{array}$

Are or not isomorphic the algebras $T4_1$ and $T4_2$? In the basis $(f_1 = -e_1, f_2 = e_2, f_3 = -e_3)$ the algebra $T4_2$ has the table $T4_1$, so that the two algebras are isomorphic.

Algebra $A(0,0,1,1)$ has the properties:

- $Ann\ A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1$, $\mathcal{I}(A) = \emptyset$,
- it is a simple algebra.

Case 5°

The algebra $A(0,b,0,0)$ with $b \neq 0$ is isomorphic with $A(0,1,0,0)$. Indeed, in basis $(\frac{1}{b}e_1, e_2, e_3)$ the multiplication table is:

$$\textbf{Table T5} \quad \begin{array}{l} e_1 \cdot e_2 = e_2, \ \ e_1 \cdot e_3 = e_3, \\ e_1^2 = e_2 \cdot e_3 = e_2^2 = e_3^2 = 0. \end{array}$$

Algebra $A(0,1,0,0)$ has the properties:

- $Ann\ A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup Span_{\mathbb{R}}\{e_2, e_3\}$, $\mathcal{I}(A) = \emptyset$,
- it is not a simple algebra ($Span_{\mathbb{R}}\{e_2,e_3\}$ is an ideal); more exactly, A has the WEDDERBURN-ARTIN decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2,e_3\}$.

Case 6°

The algebra $A(0,b,0,\varepsilon)$ with $b\varepsilon \neq 0$ is isomorphic with $A(0,1,0,\pm 1)$, depending on either $b\varepsilon$ is positive or negative. In the basis (e_1,e_2,e_3) the algebra $A(0,b,0,\varepsilon)$ has the multiplication table

$$\textbf{Table T6} \quad \begin{array}{l} e_1 \cdot e_2 = be_2, \ \ e_1 \cdot e_3 = be_3, \\ e_1^2 = e_2 \cdot e_3 = 0, \ \ e_2^2 = e_3^2 = \varepsilon e_1. \end{array}$$

Case $b\varepsilon > 0$. By passing to the basis $\left(\frac{1}{b}e_1, \frac{1}{\sqrt{b\varepsilon}}e_2, \frac{1}{\sqrt{b\varepsilon}}e_3\right)$ the multiplication table becomes:

$$\textbf{Table T6}_1 \quad \begin{array}{l} e_1 \cdot e_2 = e_2, \ \ e_1 \cdot e_3 = e_3, \\ e_1^2 = e_2 \cdot e_3 = 0, \ \ e_2^2 = e_3^2 = e_1. \end{array}$$

Case $b\varepsilon < 0$. In basis $\left(\frac{1}{b}e_1, \frac{1}{\sqrt{-b\varepsilon}}e_2, \frac{1}{\sqrt{-b\varepsilon}}e_3\right)$ the multiplication table becomes:

$$\textbf{Table T6}_2 \quad \begin{array}{l} e_1 \cdot e_2 = e_2, \ \ e_1 \cdot e_3 = e_3, \\ e_1^2 = e_2 \cdot e_3 = 0, \ \ e_2^2 = e_3^2 = -e_1. \end{array}$$

Are or not isomorphic $T6_1$ and $T6_2$? Since $T6_1$ has at lest an idempotent element (for example, $e = \frac{1}{2}(e_1 + e_2 + e_3)$), while $T6_2$ has no idempotent it results that the two algebras are not isomorphic.

Algebra $A(0,1,0,1)$ has the properties:

- $Ann\ A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1$,
  $\mathcal{I}(A) = \{\frac{1}{2}(e_1 + \sqrt{2}(e_2 cos\ \varphi) + e_3 sin\ \varphi) | \varphi \in [0, 2\pi)\}$,
- it is a simple algebra.

Algebra $A(0,1,0,-1)$ has the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1$, $\mathcal{I}(A) = \emptyset$,
• it is a simple algebra.

Case 7°

The algebra $A(0,b,c,0)$ with $bc \neq 0$ has, in basis (e_1,e_2,e_3), the multiplication table

$$\textbf{Table 7°} \quad \begin{array}{l} e_1 \cdot e_2 = be_2 + ce_3, \quad e_1 \cdot e_3 = -ce_2 + be_3, \\ e_1^2 = e_2^2 = e_3^2 = e_2 \cdot e_3 = 0. \end{array}$$

In the basis $(\frac{1}{c}e_1, e_2, e_3)$ the multiplication table becomes

$$\textbf{Table 7} \quad \begin{array}{l} e_1 \cdot e_2 = \lambda e_2 + e_3, \quad e_1 \cdot e_3 = -e_2 + \lambda e_3, \\ e_1^2 = e_2^2 = e_3^2 = e_2 \cdot e_3 = 0. \end{array}$$

with $\lambda = \dfrac{b}{c} \in \mathbb{R}^*$. Consequently, the algebra $A(0,b,c,0)$ with $bc \neq 0$ is isomorphic with the algebra $A(0,\lambda,1,0)$ with $\lambda = \dfrac{b}{c} \neq 0$. Further, by using the basis change $(e_1,e_2,e_3) \to (-e_1,e_2,-e_3)$ it results that $A(0,\lambda,1,0)$ is isomorphic with $A(0,-\lambda,1,0)$. Consequently, it is enough to study the algebras $A(0,\lambda,1,0)$ with $\lambda > 0$, only.

Algebra $A(0,\lambda,1,0)$ has the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup Span_{\mathbb{R}}\{e_2,e_3\}$, $\mathcal{I}(A) = \emptyset$,
• it is not a simple algebra; indeed, it has the WEDDERBURN-ARTIN direct sum decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2,e_3\}$,
• the following result holds:

Proposition 3.3 *The algebras $A(0,\lambda_1,1,0)$ and $A(0,\lambda_2,1,0)$ (with $\lambda_1 > 0$ and $\lambda_2 > 0$) are isomorphic if and only if $\lambda_1 = \lambda_2$.*

Case 8°

The algebra $A(0,b,c,\varepsilon)$ with $bc\varepsilon \neq 0$ has, in the basis (e_1,e_2,e_3), the multiplication table

$$\textbf{Table 8°} \quad \begin{array}{l} e_1 \cdot e_2 = be_2 + ce_3, \quad e_1 \cdot e_3 = -ce_2 + be_3, \\ e_1^2 = e_2 \cdot e_3 = 0, \quad e_2^2 = e_3^2 = \varepsilon e_1. \end{array}$$

By passing to the basis $B' = (f_1 = \varepsilon e_1, f_2 = e_2, f_3 = e_3)$ the multiplication table of the algebra becomes

$$\textbf{Table 8'} \quad \begin{array}{l} e_1 \cdot e_2 = be_2 + ce_3, \quad e_1 \cdot e_3 = -ce_2 + be_3, \\ e_1^2 = e_2 \cdot e_3 = 0, \quad e_2^2 = e_3^2 = e_1. \end{array}$$

with $b, c \in \mathbb{R}^*$ (here, it was put b instead of εb and c instead of εb).

Moreover, $c(\neq 0)$ can be always chosen positive. Indeed, if $c < 0$, in the basis $B' = (f_1 = e_1, f_2 = e_2, f_3 = -e_3)$ one gets the multiplication table

Table 8* $f_1 \cdot f_2 = bf_2 + cf_3, \quad f_1 \cdot f_3 = -cf_2 + bf_3,$
$f_1^2 = f_2 \cdot f_3 = 0, \quad f_2^2 = f_3^2 = \varepsilon f_1.$

i.e. c pass in $-c(> 0)$. Thus, $A(0, b, c, 1)$ is isomorphic with $A(0, b, -c, 1)$ and, consequently, only the study of algebras $A(0, b, c, 1)$ with $c > 0$ will be of interest.

Given an algebra $A(0, b, c > 0, 1)$ by table T8', by passing to the basis $B" = (f_1 = \frac{1}{c}e_1, f_2 = \frac{1}{\sqrt{c}}e_2, f_3 = \frac{1}{\sqrt{c}}e_3)$ the multiplication table becomes:

Table 8" $f_1 \cdot f_2 = \lambda f_2 + f_3, \quad f_1 \cdot f_3 = -f_2 + \lambda f_3,$
$f_1^2 = f_2 \cdot f_3 = 0, \quad f_2^2 = f_3^2 = f_1.$

with $\lambda \in \mathbb{R}^*$. Consequently, the multiplication table of such an algebra depends on a single parameter $\lambda = \frac{b}{c} \neq 0$.

Algebra $A(0, \lambda, 1, 1)$ $(\lambda \neq 0)$ has the properties:

- $Ann\ A = \{0\}, \quad \mathcal{N}(A) = \mathbb{R}e_1, \quad \mathcal{I}(A) = \emptyset,$
- is a simple algebra,
- the following result holds:

Proposition 3.4 *The algebras* $A(0, \lambda_1, 1, 1)$ *and* $A(0, \lambda_2, 1, 1)$ *(with* $\lambda_1\lambda_2 \neq 0$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$.

Case 9°

The algebra $A(a, 0, 0, 0)$ with $a \neq 0$ is isomorphic with the algebra $A(1, 0, 0, 0)$. This assertion can be proved by passing from basis $B = (e_1, e_2, e_3)$ to the basis $B' = (\frac{1}{a}e_1, e_2, e_3)$. Indeed, in basis B' the algebra A has the multiplication table

Table 9 $e_1^2 = e_1,$
$e_1 \cdot e_2 = e_1 \cdot e_3 = e_2^2 = e_3^2 = e_2 \cdot e_3 = 0.$

Algebra $A(1, 0, 0, 0)$ has the properties:

- $Ann\ A = Span_{\mathbb{R}}\{e_2, e_3\}, \quad \mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\}, \quad \mathcal{I}(A) = \{e_1\},$
- it is not a simple algebra; indeed, it has the WEDDERBURN-ARTIN direct sum decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$.

Case 10°

The algebra $A(a, 0, 0, \varepsilon)$ with $a\varepsilon \neq 0$ has in the basis $B = (e_1, e_2, e_3)$ the multiplication table

Table 10 $e_1^2 = ae_1, \quad e_2^2 = e_3^2 = \varepsilon e_1,$
$e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0.$

By passing from the basis $B = (e_1, e_2, e_3)$ to the basis $B' = \left(\frac{1}{a}e_1, \frac{1}{\sqrt{|a\varepsilon|}}e_2, \frac{1}{\sqrt{|a\varepsilon|}}e_3\right)$, it results that $A(a, 0, 0, \varepsilon)$ is isomorphic either with $A(1, 0, 0, 1)$ or with $A(1, 0, 0, -1)$ depending on either $a\varepsilon > 0$ or $a\varepsilon < 0$, i.e. one gets the multiplication tables

Table 10' $\quad e_1^2 = e_1,\ \ e_2^2 = e_3^2 = e_1,$
$e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0.$

and

Table 10" $\quad e_1^2 = e_1,\ \ e_2^2 = e_3^2 = -e_1,$
$e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0.$

Algebra $A(1, 0, 0, 1)$ has the properties:

- $Ann\ A = \{0\}$, $\mathcal{N}(A) = \{0\}$, $\mathcal{I}(A) = \{e_1\}$,
- it is not a simple algebra; indeed, it has the nonzero ideal $A^2 = \mathbb{R}e_1$.

Algebra $A(1, 0, 0, -1)$ has the properties:

- $Ann\ A = \{0\}$, $\mathcal{N}(A) = \{r(e_1 + e_2 cos\ \varphi + e_3 sin\ \varphi | \varphi \in [0, \pi), r > 0\}$, $\mathcal{I}(A) = \{e_1\}$,
- it is not a simple algebra; indeed, it has the nonzero ideal $A^2 = \mathbb{R}e_1$.

Since the algebra $T10$" has the nilpotent elements, while the algebra $T10'$ has no nilpotent, it results that these two algebras cannot be isomorphic.

Case 11°

The algebra $A(a, 0, c, 0)$ with $ac \neq 0$ has in the basis $B = (e_1, e_2, e_3)$ the multiplication table

Table T11° $\quad e_1^2 = ae_1,\ \ e_1 \cdot e_2 = ce_3,\ \ e_1 \cdot e_3 = -ce_2,$
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0.$

By passing from the basis $B = (e_1, e_2, e_3)$ to the basis $B' = (\frac{1}{a}e_1, e_2, e_3)$ one gets the multiplication table

Table T11* $\quad e_1^2 = e_1,\ \ e_1 \cdot e_2 = \lambda e_3,\ \ e_1 \cdot e_3 = -\lambda e_2,$
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0.$

with $\lambda = \frac{c}{a} \in \mathbb{R}^*$. Consequently, the algebra $A(a, 0, c, 0)$ is isomorphic with $A(1, 0, \lambda, 0)$ with $\lambda = \frac{c}{a} \neq 0$. It follows that the algebras $A(1, 0, \lambda, 0)$ and $A(1, 0, -\lambda, 0)$ are isomorphic; indeed, $(e_1 \to e_1, e_2 \to -e_2, e_3 \to e_3)$ yields the expected isomorphism. Consequently, it is enough to analyze only the algebras $A(1, 0, \lambda, 0)$ with $\lambda > 0$.

Algebra $A(1, 0, \lambda, 0)$ $(\lambda > 0)$ has the properties:

- $Ann\ A = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\}$, $\mathcal{I}(A) = \{e_1\}$,

• it is not a simple algebra; indeed, it has the WEDDERBURN-ARTIN direct sum decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$.

• the following result holds:

Proposition 3.5 *The algebras* $A(1,0,\lambda_1,0)$ *and* $A(0,\lambda_2,1,0)$ *(with* $\lambda_1 > 0$ *and* $\lambda_2 > 0$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$.

Case 12°

The algebra $A(a,0,c,\varepsilon)$ with $ac\varepsilon \neq 0$ has in basis $B = (e_1, e_2, e_3)$ the multiplication table

Table T12° $\quad e_1^2 = ae_1,\ \ e_1 \cdot e_2 = ce_3,\ \ e_1 \cdot e_3 = -ce_2,$
$e_2^2 = e_3^2 = \varepsilon e_1,\ \ e_2 \cdot e_3 = 0.$

In the basis $\left(\frac{1}{a}e_1, \frac{1}{\sqrt{|a\varepsilon|}}e_2, \frac{1}{\sqrt{|a\varepsilon|}}e_3\right)$ the table $T12°$ becomes

Table T12' $\quad e_1^2 = e_1,\ \ e_1 \cdot e_2 = \lambda e_3,\ \ e_1 \cdot e_3 = -\lambda e_2,$
$e_2^2 = e_3^2 = e_1,\ \ e_2 \cdot e_3 = 0,$

or

Table T12" $\quad e_1^2 = e_1,\ \ e_1 \cdot e_2 = \lambda e_3,\ \ e_1 \cdot e_3 = -\lambda e_2,$
$e_2^2 = e_3^2 = -e_1,\ \ e_2 \cdot e_3 = 0,$

depending on either $a\varepsilon > 0$ or $a\varepsilon < 0$, with $\lambda \in \mathbb{R}^*$ (it was put λ instead of $\frac{\lambda}{a}$). It can be considered that $\lambda > 0$; indeed, by passing from the basis $B = (e_1, e_2, e_3)$ to the basis $B' = (e_1, -e_2, e_3)$ the algebra T12' becomes $A(1,0,-\lambda,1)$ while the algebra T12" becomes $A(1,0,-\lambda,-1)$.

Algebras $A(1,0,\lambda,1)$ and $A(1,0,\lambda,-1)$ ($\lambda > 0$) have the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = \{0\}$, $\mathcal{I}(A) = \{e_1\}$,
• they are simple algebras.
• the following result holds:

Proposition 3.6 1) *The algebras* $A(1,0,\lambda_1,1)$ *and* $A(1,0,\lambda_2,1)$ (*with* $\lambda_1 > 0$ *and* $\lambda_2 > 0$) *are isomorphic if and only if* $\lambda_1 = \lambda_2$.

2) *The algebras* $A(1,0,\lambda_1,-1)$ *and* $A(1,0,\lambda_2,-1)$ (*with* $\lambda_1 > 0$ *and* $\lambda_2 > 0$) *are isomorphic if and only if* $\lambda_1 = \lambda_2$.

3) *Two algebras* $A(1,0,\lambda_1,1)$ *and* $A(1,0,\lambda_2,-1)$ (*with* $\lambda_1 > 0$ *and* $\lambda_2 > 0$) *are not isomorphic whatsoever be the positive values of* λ_1 *and* λ_2.

Case 13°

The algebra $A(a,b,0,0)$ with $ab \neq 0$ has in the basis $B = (e_1, e_2, e_3)$ the multiplication table

Table T13° $e_1^2 = ae_1, \;\; e_1 \cdot e_2 = be_2, \;\; e_1 \cdot e_3 = be_3,$
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0,$

The basis $(\frac{1}{a}e_1, e_2, e_3)$ assures us that the algebra $A(a, b, 0, 0)$ with $ab \neq 0$ is isomorphic with the algebra $A(1, \lambda = b/a, 0, 0)$. Let us consider an algebra $A(1, \lambda, 0, 0)$ with $\lambda \neq 0$ having the multiplication table

Table T13 $e_1^2 = e_1, \;\; e_1 \cdot e_2 = \lambda e_2, \;\; e_1 \cdot e_3 = \lambda e_3,$
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0,$

Algebra $A(1, \lambda, 0, 0)$ $(b \notin \{0, \frac{1}{2}\})$ has the properties:

• $Ann\ A = \{0\}, \;\; \mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\}, \;\; \mathcal{I}(A) = \mathbb{R}e_1$
• it is not a simple algebra; it has the WEDDERBURN-ARTIN direct sum decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$.

Algebra $A(1, \frac{1}{2}, 0, 0)$ has the properties:

• $Ann\ A = \{0\}, \;\; \mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\},$
$\mathcal{I}(A) = \mathbb{R}e_1 \cup \{e_1 + ye_2 + ze_3 | y, z \in \mathbb{R}\},$
• it is not a simple algebra; it has the WEDDERBURN-ARTIN direct sum decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$,
• the following result holds:

Proposition 3.7 *The algebras $A(1, \lambda_1, 0, 0)$ and $A(1, \lambda_2, 0, 0)$ with $\lambda_1, \lambda_2 \notin \{0, \frac{1}{2}\}$ are isomorphic if and only if $\lambda_1 = \lambda_2$.*

Case 14°

The algebra $A(a, b, 0, \varepsilon)$ with $ab\varepsilon \neq 0$ has in the basis $B = (e_1, e_2, e_3)$ the multiplication table

Table T14° $e_1^2 = ae_1, \;\; e_1 \cdot e_2 = be_2, \;\; e_1 \cdot e_3 = be_3,$
$e_2^2 = e_3^2 = \varepsilon e_1, \;\; e_2 \cdot e_3 = 0,$

By using the basis $\left(\frac{1}{a}e_1, \frac{1}{\sqrt{|a\varepsilon|}}e_2, \frac{1}{\sqrt{|a\varepsilon|}}e_3\right)$ one gets that $A(a, b, 0, \varepsilon)$ is isomorphic with one of the algebras $A(1, \lambda, 0, 1)$ and $A(1, \lambda, 0, -1)$, depending on either $a\varepsilon > 0$ or $a\varepsilon < 0$, i.e. it has either the multiplication table

Table T14' $e_1^2 = e_1, \;\; e_1 \cdot e_2 = \lambda e_2, \;\; e_1 \cdot e_3 = \lambda e_3,$
$e_2^2 = e_3^2 = e_1, \;\; e_2 \cdot e_3 = 0,$

or

Table T14" $e_1^2 = e_1, \;\; e_1 \cdot e_2 = \lambda e_2, \;\; e_1 \cdot e_3 = \lambda e_3,$
$e_2^2 = e_3^2 = -e_1, \;\; e_2 \cdot e_3 = 0,$

with $b \in \mathbb{R}^*$.

Algebra $A(1, \lambda, 0, 1)$ $(\lambda \neq 0)$ has the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = \{0\}$, $\mathcal{I}(A) = \mathbb{R}e_1$ if $\lambda \leq \frac{1}{2}$ and
$\mathcal{I}(A) = \left\{ \frac{1}{2\lambda}[e_1 + \sqrt{2\lambda - 1}(e_2 cos\ \theta + e_3 sin\ \theta] | \theta \in [0, 2\pi) \right\}$ if $\lambda > \frac{1}{2}$,

• it is a simple algebra.

Algebra $A(1, \lambda, 0, -1)$ $(\lambda \neq 0)$ has the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = \{0\}$, $\mathcal{I}(A) = \mathbb{R}e_1$ if $\lambda \geq \frac{1}{2}$ and
$\mathcal{I}(A) = \left\{ \frac{1}{2\lambda}[e_1 + \sqrt{2\lambda - 1}(e_2 cos\ \theta + e_3 sin\ \theta] | \theta \in [0, 2\pi) \right\} \cup \left\{ \frac{1}{2\lambda}[e_1 + \sqrt{2\lambda - 1}e_3] \right\}$ if $\lambda < \frac{1}{2}$,

• it is a simple algebra.

• the following result holds:

Proposition 3.8 1) *The algebras* $A(1, \lambda_1, 0, 1)$ *and* $A(1, \lambda_2, 0, 1)$ *(with* $\lambda_1, \lambda_2 \in \mathbb{R}^*$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$.

2) *The algebras* $A(1, \lambda_1, 0, -1)$ *and* $A(1, \lambda_2, 0, -1)$ *(with* $\lambda_1, \lambda_2 \in \mathbb{R}^*$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$.

3) *Two algebras* $A(1, \lambda_1, 0, 1)$ *and* $A(1, \lambda_2, 0, -1)$ *(with* $\lambda_1, \lambda_2 \in \mathbb{R}^*$*) are not isomorphic whatsoever be the allowed values of* λ_1 *and* λ_2.

Case 15°

The algebra $A(a, b, c, 0)$ with $abc \neq 0$ has in the basis $B = (e_1, e_2, e_3)$ the multiplication table

Table T15° $e_1^2 = ae_1$, $e_1 \cdot e_2 = be_2 + ce_3$, $e_1 \cdot e_3 = -ce_2 + be_3$,
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0$,

The algebra $A(a, b, c, 0)$ with $abc \neq 0$ is isomorphic with the algebra $A(1, b/a, c/a, 0)$; it is enough to consider the basis $(\frac{1}{a}e_1, e_2, e_3)$. Let us consider the algebra $A(1, b, c, 0)$ with $bc \neq 0$ having the multiplication table

Table T15 $e_1^2 = e_1$, $e_1 \cdot e_2 = be_2 + ce_3$, $e_1 \cdot e_3 = -ce_2 + be_3$,
$e_2^2 = e_3^2 = e_2 \cdot e_3 = 0$,

The algebra $A(a, b, c, 0)$ is isomorphic with the algebra $A(a, b, -c, 0)$; indeed, the forecasted isomorphism is

$$e_1 \to e_1, e_2 \to e_2, e_3 \to -e_3.$$

Consequently, in the followings we shall consider only algebras $A(1, \lambda, \mu, 0)$ with $\lambda \neq 0$ and $\mu > 0$.

Algebra $A(1, \lambda, \mu, 0)$ $(\lambda \neq 0,\ \mu > 0)$ has the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_2, e_3\}$, $\mathcal{I}(A) = \{e_1\}$,
• it is not a simple algebra; indeed, it has the WEDDERBURN-ARTIN direct sum decomposition $A = \mathbb{R}e_1 \oplus Span_{\mathbb{R}}\{e_2, e_3\}$.
• the following result holds:

Proposition 3.9 *The algebras* $A(1, \lambda_1, \mu_1, 0)$ *and* $A(1, \lambda_2, \mu_2, 0)$ *(with* $\mu_1, \nu_2 > 0$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$.

Further, it must be remarked that $A(\kappa, \omega)$ is coincident with $A(1, \kappa, -\omega, 0)$.

Case 16°

The algebra $A(a, b, c, \varepsilon)$ with $abc\varepsilon \neq 0$ has in the basis $B = (e_1, e_2, e_3)$ the multiplication table

Table T16° $\quad e_1^2 = ae_1,\ e_1 \cdot e_2 = be_2 + ce_3,\ e_1 \cdot e_3 = -ce_2 + be_3,$
$e_2^2 = e_3^2 = \varepsilon e_1,\ e_2 \cdot e_3 = 0,$

By using the basis $\left(\frac{1}{a}e_1, \frac{1}{\sqrt{|a\varepsilon|}}e_2, \frac{1}{\sqrt{|a\varepsilon|}}e_3\right)$ one proves that $A(a, b, c, \varepsilon)$ is isomorphic with one of the algebras $A(1, \lambda, \mu, 1)$ and $A(1, \lambda, \mu, -1)$ with $b, c \in \mathbb{R}^*$, depending on either $a\varepsilon > 0$ or $a\varepsilon < 0$, i.e. with algebras having the multiplication tables

Table T16' $\quad e_1^2 = e_1,\ e_1 \cdot e_2 = \lambda e_2 + \mu e_3,\ e_1 \cdot e_3 = -\mu e_2 + \lambda e_3,$
$e_2^2 = e_3^2 = e_1,\ e_2 \cdot e_3 = 0,$

Table T16" $\quad e_1^2 = e_1,\ e_1 \cdot e_2 = \lambda e_2 + \mu e_3,\ e_1 \cdot e_3 = -\mu e_2 + \lambda e_3,$
$e_2^2 = e_3^2 = -e_1,\ e_2 \cdot e_3 = 0,$

with $b_1, c_1, b_2, c_2 \in \mathbb{R}^*$. By using the basis $(e_1, e_2, -e_3)$ one proves that $A(1, \lambda, \mu, \pm 1)$ is isomorphic with the algebra $A(1, \lambda, -\mu, \pm 1)$. Consequently, in the followings we shall study only algebras $A(1, \lambda, \mu, \pm 1)$ with $\lambda \in \mathbb{R}^*$ and $\mu > 0$.

Algebra $A(1, \lambda, \mu, \pm 1)$ (with $\lambda \in \mathbb{R}^*$ and $\nu > 0$) has the properties:

• $Ann\ A = \{0\}$, $\mathcal{N}(A) = \{0\}$, $\mathcal{I}(A) = \{e_1\}$,
• it is a simple algebra.
• the following result holds:

Proposition 3.10 1) *The algebras* $A(1, \lambda_1, \mu_1, 1)$ *and* $A(1, \lambda_2, \mu_2, 1)$ *(with* $\lambda_1, \lambda_2 \in \mathbb{R}^*, \mu_1, \mu_2 > 0$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$.

2) *The algebras* $A(1, \lambda_1, \mu_1, -1)$ *and* $A(1, \lambda_2, \mu_2, -1)$ *(with* $\lambda_1, \lambda_2 \in \mathbb{R}^*, \mu_1, \mu_2 > 0$*) are isomorphic if and only if* $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$.

3) *Two algebras* $A(1, \lambda_1, \mu_1, 1)$ *and* $A(1, \lambda_2, \mu_2, -1)$ *(with* $\lambda_1, \lambda_2 \in \mathbb{R}^*, \mu_1, \mu_2 > 0$*) are not isomorphic whatsoever be the allowed values of* $\lambda_1, \lambda_2, \mu_1$ *and* μ_2.

By using the exhibited properties and Proposition 2.1 it results that any two algebras cannot be isomorphic each other.

Theorem 3.11 *On $\mathbb{R}^3$ there exist 20 classes of nonzero commutative algebras, having at least a derivation with a complex eigenvalue, nonisomorphic each other, namely:* $A(0,0,0,1)$, $A(0,0,1,0)$, $A(0,0,1,1)$, $A(0,1,0,0)$, $A(0,1,0,1)$, $A(0,1,0,-1)$, $A(0,\lambda,1,0)$ $(\lambda>0)$, $A(0,\lambda,1,1)$ $(\lambda\neq 0)$, $A(1,0,0,0)$, $A(1,0,0,1)$, $A(1,0,0,-1)$, $A(1,0,\lambda,0)$ $(\lambda>0)$, $A(1,0,\lambda,1)$ $(\lambda>0)$, $A(1,0,\lambda,-1)$ $(\lambda>0)$, $A(1,\lambda,0,0)$ $(\lambda\notin\{0,\frac{1}{2}\})$, $A(1,\lambda,0,1)$ $(\lambda\neq 0)$, $A(1,\lambda,0,-1)$ $(\lambda\neq 0)$, $A(1,\lambda,\mu,0)$ $(\lambda\neq 0,\ \mu>0)$, $A(1,\lambda,\mu,1)$ $(\lambda\mu\neq 0)$, $A(1,\lambda,\mu,-1)$ $(\lambda\mu\neq 0)$.

Consequently, it was proved the result:

Theorem 3.12 *For any nontrivial HQDS on $\mathbb{R}^3$ there exists a center-affinity such that it becomes one of the following 20 HQDSs:*

$$1^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^2)^2+(x^3)^2\\ \dfrac{dx^2}{dt}=0\\ \dfrac{dx^3}{dt}=0,\end{cases}\qquad 2^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=0\\ \dfrac{dx^2}{dt}=-2x^1x^3\\ \dfrac{dx^3}{dt}=\ \ 2x^1x^2,\end{cases}$$

$$3^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^2)^2+(x^3)^2\\ \dfrac{dx^2}{dt}=-2x^1x^3\\ \dfrac{dx^3}{dt}=\ \ 2x^1x^2,\end{cases}\qquad 4^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=0\\ \dfrac{dx^2}{dt}=2x^1x^2\\ \dfrac{dx^3}{dt}=2x^1x^3,\end{cases}$$

$$5^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^2)^2+(x^3)^2\\ \dfrac{dx^2}{dt}=2x^1x^2\\ \dfrac{dx^3}{dt}=2x^1x^3,\end{cases}\qquad 6^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=-(x^2)^2-(x^3)^2\\ \dfrac{dx^2}{dt}=2x^1x^2\\ \dfrac{dx^3}{dt}=2x^1x^3,\end{cases}$$

$$7^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=0\\ \dfrac{dx^2}{dt}=2\lambda x^1x^2-2x^1x^3\\ \dfrac{dx^3}{dt}=2x^1x^2+2\lambda x^1x^3,\\ (\lambda>0)\end{cases}\qquad 8^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^2)^2+(x^3)^2\\ \dfrac{dx^2}{dt}=2\lambda x^1x^2-2x^1x^3\\ \dfrac{dx^3}{dt}=2x^1x^2+2\lambda x^1x^3,\\ (\lambda\neq 0)\end{cases}$$

$$9^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^1)^2\\ \dfrac{dx^2}{dt}=0\\ \dfrac{dx^3}{dt}=0,\end{cases}\qquad 10^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^1)^2+(x^2)^2+(x^3)^2\\ \dfrac{dx^2}{dt}=0\\ \dfrac{dx^3}{dt}=0,\end{cases}$$

$$11^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=-(x^1)^2-(x^2)^2-(x^3)^2\\ \dfrac{dx^2}{dt}=0\\ \dfrac{dx^3}{dt}=0,\end{cases}\qquad 12^\circ.\ \begin{cases}\dfrac{dx^1}{dt}=(x^1)^2\\ \dfrac{dx^2}{dt}=-2\lambda x^1x^3\\ \dfrac{dx^3}{dt}=2\lambda x^1x^2,\\ (\lambda>0)\end{cases}$$

$$13^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 + (x^2)^2 + (x^3)^2 \\ \dfrac{dx^2}{dt} = -2\lambda x^1 x^3 \\ \dfrac{dx^3}{dt} = 2\lambda x^1 x^2, \end{cases} \quad (\lambda > 0)$$

$$14^\circ.\begin{cases} \dfrac{dx^1}{dt} = -(x^1)^2 - (x^2)^2 - (x^3)^2 \\ \dfrac{dx^2}{dt} = -2\lambda x^1 x^3 \\ \dfrac{dx^3}{dt} = 2\lambda x^1 x^2, \end{cases} \quad (\lambda > 0)$$

$$15^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 \\ \dfrac{dx^2}{dt} = 2\lambda x^1 x^2 \\ \dfrac{dx^3}{dt} = 2\lambda x^1 x^3, \end{cases} \quad (\lambda \notin \{0, \tfrac{1}{2}\})$$

$$16^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 + (x^2)^2 + (x^3)^2 \\ \dfrac{dx^2}{dt} = 2\lambda x^1 x^2 \\ \dfrac{dx^3}{dt} = 2\lambda x^1 x^3, \end{cases} \quad (\lambda \neq 0)$$

$$17^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 - (x^2)^2 - (x^3)^2 \\ \dfrac{dx^2}{dt} = 2\lambda x^1 x^2 \\ \dfrac{dx^3}{dt} = 2\lambda x^1 x^3, \end{cases} \quad (\lambda \neq 0)$$

$$18^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 \\ \dfrac{dx^2}{dt} = 2\lambda x^1 x^2 - 2\mu x^1 x^3 \\ \dfrac{dx^3}{dt} = 2\mu x^1 x^2 + 2\lambda x^1 x^3, \end{cases} \quad (\lambda \neq 0, \mu > 0)$$

$$19^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 + (x^2)^2 + (x^3)^2 \\ \dfrac{dx^2}{dt} = 2\lambda x^1 x^2 - 2\mu x^1 x^3 \\ \dfrac{dx^3}{dt} = 2\mu x^1 x^2 + 2\lambda x^1 x^3, \end{cases} \quad (\lambda\mu \neq 0)$$

$$20^\circ.\begin{cases} \dfrac{dx^1}{dt} = (x^1)^2 - (x^2)^2 - (x^3)^2 \\ \dfrac{dx^2}{dt} = 2\lambda x^1 x^2 - 2\mu x^1 x^3 \\ \dfrac{dx^3}{dt} = 2\mu x^1 x^2 + 2\lambda x^1 x^3 \end{cases} \quad (\lambda\mu \neq 0).$$

References

[1] I. Burdujan, *Quadratic differential systems* (Romanian), Ed. PIM, 2008.

[2] I. Burdujan, *A classification of a class of homogeneous quadratic dynamical systems on* $\mathbb{R}^3$ *with derivations*, (in press), Bull. I.P. Iaşi.

[3] I. Burdujan, *n-Homogeneous dynamical systems and n-ary algebras*, Bul. Acad. St. Rep. Moldova, no. 2(54), 2007, 1–15.

[4] N. Dumitriu, I. Burdujan, G. Dimitriu, *Mathematical Models in Biology and Environmental Sciences* (Romanian), Ed. CERMI, Iaşi, 2005.

[5] A.A. Sagle, R. Walde, *Introduction to Lie groups and Lie algebras*, Academic Press, New York, 1973.