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# A Physical Approach to a Class of Diophantine Equations

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**Abstract.** In this work, we introduce and specify how to apply physics to solve or estimate some problems in number theory. More precisely, we use center of mass (centroid) to show how to find or estimate sum of powers of distinct positive numbers (integers) with fixed positive coefficients. Consequently, as a corollary, we write an alternative expression of Fermat's Last Theorem for positive integers and predict some impossible cases. That is, for any fixed positive integers $x < y$, there always exists a sufficiently large positive integer $k_0$ such that $x^k + y^k = z^k$ is impossible for all positive integers $z$ and all $k \geq k_0$. Moreover, we show that $k \geq \frac{ln2}{ln(y+1)-lny}$.

**Keywords:** $k$-Mass-System of $n$ points (integers), center of mass (centroid).

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## Arithmetic via Center of Mass

We usually in physics apply mathematics to solve or express a physical phenomenon by finding a mathematical model (expression). In this work, our goal is to show the capability of physics to solve (approximate) a mathematical problem (expression) via center of mass. For more discussion and a general overview of this matter, see [4]. Also, author assumes that the reader is familiar with the concept of center of mass or centroid; and the problems in number theory that we use in this article. For a detailed study of center of mass, the reader is referred to [2] and [6]; and for the related problems in number theory, see [1], [3], and [5].

Next, in order to justify the concept of this subject, we start with two simple examples from number theory as follows.

**Example 1.** *What is the sum $S = 1 + 2 + \cdots + n$? Suppose each number is a physical point having one unit of mass and $x_i = i$ is the distance of the ith point (number) from the origin for each $i = 1, 2, \ldots, n$. Hence, by the definition of centroid, $S = \sum_1^n m_i x_i = c \sum_1^n m_i$ implies $S = cn$, where $c = (n + 1)/2$ is the center of mass of the n points on x-axis which is precisely in the middle of the interval $[1, n]$ since all the points in the mass system have equal masses and distributed uniformly; and $m_i = 1$ unit of mass for each $i = 1, 2, \ldots, n$.*

**Example 2.** *Let $x_1, x_2, \ldots, x_n$ be a strictly increasing finite sequence of positive integers. What is $S = \sum_1^n x_i$? Again, we assume each number $x_i$ is a physical point with mass $m_i = 1$*

*for each $i = 1, 2, \ldots, n$. Now, we apply the above centroid method and get $S = cn$, where $c$ is the center of mass of the system.*

**Remark 1.** *Note that in a mass system of points, the center of mass or equilibrium point of the system is closer to the heavier (massive) part of the system. Hence, any method or means that helps us to get a closer value (approximation) of the center of mass or centroid yields a better approximation of the unknown in the mathematical expression.*

**Definition 1.** *Let $n \geq 2$, $k \geq 1$, and $a_1, a_2, \ldots, a_n$ be positive fixed integers. A $k$-mass-system of $n$ points with coefficients $a_1, a_2, \ldots, a_n$, denoted by $M(k, a_1, a_2, \ldots, a_n)$, is a strictly increasing sequence $x_1 < x_2 < \cdots < x_n$ of $n$ positive integers on x-axis with each point $x_i$ having $m_i = a_i x_i^{k-1}$ unit(s) of mass for each $i = 1, 2, \ldots, n$. Note that for the sake of convenience, $M(k, a_1, a_2, \ldots, a_n)$ will simply be denoted by $M(k, n)$ whenever $a_1 = a_2 = \cdots = a_n = 1$.*

**Theorem 1.** *Let for fixed positive integers $n \geq 2$ and $k \geq 1$, $M(k, a_1, a_2, \ldots, a_n)$ be a $k$-mass-system with fixed positive coefficients $a_1, a_2, \ldots, a_n$. Let $S_k = \sum_{i=1}^{n} a_i x_i^k$. Suppose each number $x_i$ is a physical point having $m_i = a_i x_i^{k-1}$ unit(s) of mass. Also, assume $S_0 = a_1 + a_2 + \cdots + a_n$. Then $S_k = S_0 c_1 c_2 \cdots c_k$, where $c_i$ is the center of mass of the $i$-mass-system for each fixed $i = 1, 2, \ldots, k$.*

*Proof.* From the definition, it follows that $\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = c_k$ is the center of mass of $n$ points ($x_i$'s on x-axis) with $m_i$ unit(s) of mass for each $i = 1, 2, \ldots, n$. Hence, $S_k = \sum_{i=1}^{n} m_i x_i = c_k S_{k-1}$, where $c_k$ is the center of mass of the mass system $M(k, a_1, a_2, \ldots, a_n)$ with $n$ points $x_1 < x_2 < \cdots < x_n$ on x-axis. Also, note that $S_0 = a_1 + a_2 + \cdots + a_n$ by assumption. Now, from the above, it is clear that

$$\frac{S_k}{S_{k-1}} \frac{S_{k-1}}{S_{k-2}} \cdots \frac{S_1}{S_0} = c_k c_{k-1} \cdots c_1.$$

Thus, $S_k = S_0 c_1 c_2 \cdots c_k$. □

**Corollary 2.** *Suppose $x < Y$ is a mass system of 2 positive fixed integers (points) on x-axis with coefficients $a_1 = a_2 = 1$. Then for any fixed integer $k \geq 3$, the statement "$x^k + y^k = S_k = 2c_1 c_2 \cdots c_k$ can not be $k$th power of an integer" is equivalent to Fermat's Last Theorem for positive integers.*

**Remark 2.** *In the above corollary, suppose $S_k = Z^k$ for some integer $z$ and positive integer $k \geq 3$. Thus, if the product of $c_i$'s (the center of masses) is an integer, then $z$ can not be an odd integer. Furthermore, If $c_1 c_2 \cdots c_k$ is a fraction with a denominator different from 1 and 2, then the equality $S_k = z^k$ is never valid for any integer $z$. Actually, $c_1 c_2 \cdots c_k$ must always be an integer or a fraction (of course, in reduced form) with denominator 2 since $S_k$ is always a positive integer. Furthermore, in the following general case, we will find a precise lower bound for $k$.*

**Theorem 3.** *Let $n \geq 2$, $a_1, a_2, \ldots, a_n$, and $x_1 < x_2 \cdots < x_n$ be any fixed positive integers. then there exists a positive integer $k_0$ such that the Diophantine equation $a_1 x_1^k + a_2 x_2^k + \cdots + a_n x_n^k = z^k$ is never valid for all $k \geq k_0$ and any positive integer $z$. Furthermore, $k \geq \frac{lnS_0}{ln(x_n+1) - lnx_n}$, where $S_0 = a_1 + a_2 + \ldots + a_n$.*

*Proof.* From the above theorem, we have $S_k = S_0c_1c_2\cdots c_k = z^k$. Let $c = Max\{c_1, c_2, \ldots, c_k\}$. Then $S_k = a_1x_1^k + a_2x_2^k + \cdots + a_nx_n^k = z^k = S_0c_1c_2\cdots c_k \leq S_0c^k < S_0x_n^k$ implies $z < S_0^{1/k}x_n$. Therefore, for each fixed positive integer $x_n$, there always exists a sufficiently large integer $k_0$ such that $0 \leq S_0^{1/k}x_n - x_n < 1$ for all $k \geq k_0$. Hence, for sufficiently large $k$, $x_n$ is the largest integer less than $S_0^{1/k}x_n$. Consequently, $z \leq x_n$, implies $z^k \leq a_nx_n^k$. This can not happen since $a_ix_i$'s $(1 \leq i \leq n-1)$ are all positive integers. The last part of the theorem follows directly from $0 \leq S_0^{1/k}x_n - x_n \leq 1$. □

**Remark 3.** *Let $n \geq 2$, $a_1, a_2, \ldots, a_n$, and $x_1 < x_2 \cdots < x_n$ be any fixed positive integers as in the above theorem. Given a positive integer $k$and equation $a_1x_1^k + a_2x_2^k + \cdots + a_nx_n^k = z^k$. Obviously, according to the above theorem, there is no solution to this equation whenever $k \geq \frac{lnS_0}{ln(x_n+1) - lnx_n}$, where $S_0 = a_1 + a_2 + \cdots + a_n$. Now suppose $k$ is a positive integer less than $\frac{lnS_0}{ln(x_n+1) - lnx_n}$. In this case, if there exists a possible solution (a positive integer $z$) for this equation, then $z$ must satisfy the condition that $x_n \leq z \leq S_0^{1/k}x_n$ with $a_nx_n^k < z^k$. Also, from the above theorem, we can conclude the following corollary.*

**Corollary 4.** *for any fixed positive integers $x < y$, there always exists a sufficiently large positive integer $k_0$ such that $x^k + y^k = z^k$ is impossible for all positive integers $z$ and all $k \geq k_0$. More precisely, $k \geq \frac{ln2}{ln(y+1) - lny}$.*

**Remark 4.** *In the end, the Author believes that the center-of-mass method, merely, or together with a probabilistic approach could be very useful and efficient for investigation and study of Diophantine inequalities (approximations), specially for those people who like to chalenge these type of problems via a computer programming (simulation) or heuristic methods.*

# References

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