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FILIAL SEMIGROUPS AND SEMIGROUP RINGS

W. B. Vasantha Kandasamy

In this note we define what are called filial semigroups and obtain conditions for a semigroup ring RS to be filial. R. Andruszkiewicz in [1] calls a ring R to be filial if the relation ideal in R is transitive, that is if a subring J is an ideal in a subring I, and I is an ideal in R, then J is an ideal in R. For more properties of filial rings please refer [1].

Definition 1. A semigroup S is called filial if there exists at least one subsemigroup J and if J is an ideal in a subsemigroup I and I is an ideal in S then J is an ideal in S.

Example. Take $S=\{(0,0,0), (0,0,1), (0,1,0), (1,0,0)\}$ under usual componentwise multiplication $J=<(0,0,0), (0,0,1)>$, $I=<(0,0,0), (0,0,1), (0,1,0)>$ I and J are subsemigroups and J is an ideal in I and S; and I is an ideal in S. So S is filial semigroup.

It is nice to note that if S is an idempotent semigroup in which product of any distinct elements is zero, then S is filial. We say if every pair of subsemigroups satisfying the inclusion relation is transitive then S is strongly filial. Clearly the class of semigroups defined above is strongly filial.

Definition 2. Let R be a ring and S a semigroup. The semigroup ring RS is called filial if there exists at least one subring J which is an ideal in a subring I, and I is an ideal in RS, then J is an ideal of RS.

Example. Take $S = \{(0,0,0), (0,1,0), (1,0,0), (0,0,1)\}$ and $Z_2 = (0,1)$. Z_2S is a filial ring.

Remark. If every pair of subrings satisfying a inclusive relation satisfies the transitivity condition, then RS is said to be strongly filial.

Theorem 3. Let K be a field and S a filial semigroup. Then the semigroup ring KS is filial.

Proof. S be a filial semigroup, let $J \subsetneq I \subsetneq S$ then $KJ \subsetneq KI \subsetneq KS$ is such that the subring KJ is an ideal of the subring KI and KS and KI is an ideal of KS. Hence KS is filial.

Theorem 4. Let R be a filial ring. S any semigroup; then RS is a filial semigroup ring.

Proof. Given R is filial. Hence $J \subseteq I \subseteq R$ is such that the subring J is an ideal of the subring I, and I is an ideal of R, and J is an ideal R. Take $JS \subseteq IS \subseteq RS$. Clearly RS is filial.

Example. Let S be any semigroup Z ring of integers. ZS is filial $J = <4>$, $I = <2>$, $J \subsetneq I \subsetneq Z$ so $JS \subsetneq IS \subsetneq ZS$. Hence ZS is filial.

Theorem 5. Let S be a semigroup of order p and $Z_p = (0,1,2,\cdots,p-1)$, Z_pS is filial ($p - a$ prime).

Proof. Let $$\omega(Z_pS) = \left\{ \alpha = \sum \alpha_i s_i \;\middle|\; \begin{array}{c} \alpha_i \in Z_p \\ s_i \in S \\ \sum \alpha_i = 0 \end{array} \right\}$$

is the argumentation ideal of Z_pS. Let $S = (s_1, s_2, \cdots, s_p)$. let $J = \{0, n(s_1 + s_2 + \cdots + s_p) \mid 1 \leq n \leq p-1\}$, Then J is a subring and ideal of Z_pS. J is also an ideal of $\omega(Z_pS)$. So Z_pS is filial.

Problem 1. Let $Z_p = (0,1,2,\cdots,p-1)$, p s prime S a nonfilial semigroup with $|S| = n$, $(n,p) = 1$. Is Z_pS filial?

Problem 2. Let R be nonfilial ring and S a nonfilial semigroup. Can RS be filial?

REFERENCES

1. R. Andruszkiewicz, On filial rings, Portugaliae Mathematics, 45, No. 2, (1988), 136-149.